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Recursion relations for gauge theory amplitudes with massive particles

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Abstract: We derive general tree-level recursion relations for amplitudes which include massive propagating particles. As an illustration, we apply these recursion relations to scattering amplitudes of gluons coupled to massive scalars. We provide new results for all amplitudes with a pair of scalars and $n \leq 4$ gluons. These amplitudes can be used as building blocks in the computation of one-loop 6-gluon amplitudes using unitarity based methods.

Keywords: Duality in Gauge Field Theories, Supersymmetry and Duality, NLO Computations, QCD.

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Contents

1. Introduction

The twistor string description of $\mathcal{N} = 4$ Yang-Mills proposed by Witten [1] has been an inspiration for much of the recent progress in detailed calculations of multi-particle amplitudes in gauge theory. New ideas have led to the development of powerful new formalisms for these calculations, most notably the MHV rules of ref. [2], the Britto-Cachazo-Feng (BCF) recursion relations of refs. [3, 4], and the generalized unitarity cuts in the complexified Minkowski space of ref. [5]. Applications of these new formalisms together with the classic unitarity based approach of refs. [6, 7] have led to a dramatic progress in calculations of amplitudes.

At tree level, new and compact results for scattering amplitudes were derived using the MHV rules [2, 8 − 18], and the recursion relations [3, 4, 19 − 23]. External massive particles, such as Higgs bosons [14, 16] and electroweak bosons [15], have also been included using generalizations of MHV rules. On the other hand, the BCF recursion relations have been so far considered only for massless particles.

The first motivation of this paper is to construct tree-level recursion relations which naturally incorporate massive particles. The recursion relations follow from general quantum field theory arguments, and are valid for any quantum field theory (and also for gravity). The advantages of the recursion relations and the MHV rules approaches over

JHEP07(2005)025

standard Feynman-diagram calculations are obvious. Just as their massless parents [3, 4], the recursion relations for massive particles do not suffer from the factorial growth of the number of contributing Feynman diagrams. Compared to MHV rules, the recursion relations give more compact results. They can be rigorously derived for a general quantum field theory and they incorporate massive particles in a natural and universal way. Essentially, with the general recursion relations in place, one may be able to avoid in future all Feynman-diagrams calculations of nontrivial tree amplitudes.

In this paper we will use massive recursion relations to derive compact amplitudes for gluons coupled to massive coloured scalars. In the forthcoming companion paper [24] we will use the recursion relations for amplitudes with massive particles with spin.

At one loop, the MHV rules have been successfully applied to supersymmetric amplitudes [25 − 31]. New classes of amplitudes have been derived with the new methods in refs. [32 − 36, 5, 37 − 42]. All of these results give complete amplitudes only in supersymmetric theories which are cut-constructible in 4 dimensions [6, 7]. In non-supersymmetric gauge theories, the new methods apply only to the 4D cut-constructible parts of the amplitudes.

The second motivation of this paper is to assemble together the pieces necessary for the complete calculation of one-loop amplitudes in non-supersymmetric gauge theories.

One-loop multi-gluon amplitudes in the 'all-plus', and in the 'one-minus' helicity configurations, $(+, +, +, \ldots, +)$ and $(-, +, +, \ldots, +)$, are known [43, 44, 38]. However, the full one-loop amplitudes even in the simplest non-supersymmetric gauge theory, pure Yang-Mills, are still known only for $n \leq 5$ external gluons, with the five-gluon amplitude calculated in 1993 in ref. [45]. The six-gluon amplitude has not yet been calculated for all helicity configurations. One-loop amplitudes with n external gluons in pure Yang-Mills are conveniently decomposed as

$$\mathcal{A}_n^{\text{gluon}} = \mathcal{A}_n^{\mathcal{N}=4} - 4\mathcal{A}_n^{\text{chiral }\mathcal{N}=1} + \mathcal{A}_n^{\text{scalar}}\,, \tag{1.1}$$

where the first and the second terms are the contributions of the $\mathcal{N} = 4$ supermultiplet, and the chiral $\mathcal{N} = 1$ supermultiplet running in the loop. These contributions arise from supersymmetric theories, they are cut-constructible and largely known (at present all colour-ordered sub-amplitudes are known for $\mathcal{A}_n^{\mathcal{N}=4}$ for $n \leq 7$, [32, 33, 37], and the complete $\mathcal{A}_n^{\text{chiral }\mathcal{N}=1}$ is known for $n \leq 6$, [6, 7, 30, 31, 34, 35, 39, 42]).

The last term in (1.1) corresponds to an n-gluon amplitude with a complex scalar propagating in the loop. This is a non-supersymmetric one-loop amplitude and it is not cut-constructible [7]. What this means is that the amplitude $\mathcal{A}_n^{\text{scalar}}$ cannot be fully reconstructed from its imaginary part evaluated in 4 dimensions. In other words, the knowledge of the cuts of the amplitude is insufficient to recover the full answer. The answer contains purely rational terms, these do not have cuts in 4 dimensions.

To regulate infrared and ultraviolet divergent integrations over loop momenta, loop amplitudes are commonly evaluated in $D = 4 - 2\varepsilon$ dimensions. In D dimensions (with non-integer D), the amplitudes contain *only* cut-constructible contributions [46, 47]. This is because all terms in the result are proportional to D-dependent powers of the kinematic invariants, which give rise to terms like $(-s)^{-\varepsilon}$ that necessarily contain logarithms [46].

JHEP07(2005)025

Hence, in D-dimensions all amplitudes are completely determined from their cuts and are cut-constructible. The price to pay is that we now need to know on-shell tree-level amplitudes in D-dimensions and work to higher order in ε.

In order to determine full amplitude $\mathcal{A}_n^{\rm scalar}$ we cut the propagators in the loop and put them on-shell in D-dimensions. On both sides of the cut we are left with tree-level amplitudes of the form

$$\mathcal{A}_{m+2}^{\rm tree}(\phi_{l_1}^{\dagger}, g_1, g_2, \ldots, g_m, \phi_{l_2})\,. \tag{1.2}$$

These are the colour-ordered subamplitudes with two adjacent scalars and m gluons of arbitrary helicities. The external gluons have four-dimensional momenta $p_1, \ldots, p_m$. The scalar momenta l_1 and l_2 are D-dimensional, they are the loop momenta which we cut.

D-dimensional massless momenta of scalar fields can be thought of as the 4-dimensional on-shell momenta of particles with mass μ, such that $l_1^2 = l_2^2 = \mu^2$. Here l_1 and l_2 are 4-dimensional and the μ^2 term arises from the extra $D-4 = -2\varepsilon$ dimensions [46, 47].

Hence, in order to calculate 6-gluon one-loop amplitudes we need to know tree-level $4,5,6$-point amplitudes (1.2) with $m = 2,3,4$ gluons for all independent helicity arrangements. In ref. [47], the amplitudes (1.2) were calculated for up to 4 gluons with the same helicity. Using the massive recursion relation we will reproduce these results as well as derive the remaining amplitudes (1.2) with positive and negative helicity gluons.

Notation

We will be using the spinor helicity formalism [48]. A vector is represented as a bispinor

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}} p_{\mu}\ , \tag{1.3}$$

where $\sigma^{\mu}_{a\dot{a}}$ are the chiral gamma matrices. The norm of p is $p_\mu p^\mu = \det(p^{a\dot{a}})$. Hence, a vector is null if and only if it can be written as a product of two spinors

$$p^2 = 0 \Leftrightarrow p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}\,. \tag{1.4}$$

Here, λ^a and $\tilde{\lambda}^{\dot{a}}$, are commuting spinors of negative and positive helicity respectively. For real momenta in Minkowski signature $\tilde{\lambda}$ is a complex conjugate of λ, hence λ is conventionally called the "holomorphic" and $\tilde{\lambda}$ is the "anti-holomorphic" spinor. When discussing the recursion relations, it will be necessary to consider complex momenta. For complex momenta, λ^a and $\tilde{\lambda}^{\dot{a}}$ are independent complex variables.

We will often use a short-hand notation:

$$\lambda_i^a = |i\rangle^a \qquad \tilde{\lambda}_i^{\dot{a}} = |i]^{\dot{a}}\,. \tag{1.5}$$

The invariant spinor products are defined as

$$\langle i\ j\rangle \equiv \langle i^-|j^+\rangle = \lambda_i{}^a \lambda_{j\ a}\,, \qquad [i\ j] \equiv \langle i^+|j^-\rangle = \tilde{\lambda}_i{}^{\dot{a}} \tilde{\lambda}_{j\ \dot{a}}\,. \tag{1.6}$$

Here spinor indices are raised and lowered with ϵ-symbols. The scalar product of a null vector p_i with a vector q is

$$p_i \cdot q = -\frac{1}{2}\langle i|q|i]\,. \tag{1.7}$$

Throughout the paper we use the sign conventions[1] of [1 – 3] and define

$$
\begin{aligned}
s_{ij\ldots k} &= (p_i + p_j + \ldots + p_k)^2 && (1.8)\\
\langle i|p|j] &= \langle i|^a\, p_a{}^{\dot a}|j]_{\dot a} = -\lambda_i^a\, p_{a\dot a}\tilde\lambda_j^{\dot a} && (1.9)\\
\langle i|p_r p_s|j\rangle &= \langle i|^a\, p_{ra}{}^{\dot a} p_{s\dot a}{}^{b}|j\rangle_b = -\lambda_i^a p_{r\,a\dot a}\; p_s^{\dot a b}\lambda_{j\,b} && (1.10)\\
[i|p_r p_s|j] &= [i|^{\dot a}\, p_{r\dot a}{}^{a}\; p_{sa}{}^{\dot b}\,|j]_{\dot b} = -\tilde\lambda_i^{\dot a}\, p_{r\,\dot a b} p_s^{b\dot b}\tilde\lambda_{j\,\dot b}\,. && (1.11)
\end{aligned}
$$

For massless momenta equations (1.8)-(1.11) become

$$
\begin{aligned}
s_{ij} &= \langle i\, j\rangle [i\, j]\,, &\qquad \langle i|p_r|j] &= \langle ir\rangle [r\, j]\,, && (1.12)\\
\langle i|p_r p_s|j\rangle &= \langle i\, r\rangle [r\, s]\langle s\, j\rangle\,, &\qquad [i|p_r p_s|j] &= [i\, r]\langle rs\rangle [s\, j]\,, && (1.13)
\end{aligned}
$$

and so on.

2. Recursion relations with massive particles

Consider a tree level scattering amplitude of n incoming particles, some of which might be massive

$$\mathcal{A}(p_1, p_2, \ldots, p_n)\,, \qquad p_i^2 = m_i^2\,. \tag{2.1}$$

Single out two particles, i, j, for special treatment. These particles can be either massive or massless. For given p_i, p_j pick a null vector $\eta = \lambda_\eta \tilde\lambda_\eta$ that is orthogonal to both p_i and p_j

$$\eta \cdot p_i = \eta \cdot p_j = \eta^2 = 0\,. \tag{2.2}$$

For generic p_i and p_j, there are exactly two such η up to scaling. To see this, consider the plane spanned by p_i and p_j. Geometrically, the first two conditions in eq. (2.2) mean that η lies in the plane orthogonal to the plane spanned by p_i, p_j. The last condition sets η to be in the intersection of this plane with the lightcone. A generic plane going through the origin intersects the complex lightcone at two complex rays. These rays define the two solutions for η up to a scaling by a complex number. We now will construct these solutions.

Solution for the shift momentum

Let us find now explicit solutions of eqs. (2.2) for complex momenta. We will discuss in turn the case when both i, j are massless, when one of i, j is massive and at last the case when both i, j are massive.

If the marked momenta are null, $p_i = \lambda_i \tilde\lambda_i$, $p_j = \lambda_j \tilde\lambda_j$, then the condition that η is orthogonal to p_i gives $2\eta \cdot p_i = \langle \eta, i\rangle [\eta, i] = 0$, so either $\lambda_\eta = \lambda_i$ or $\tilde\lambda_\eta = \tilde\lambda_i$,. Similarly, vanishing of the Lorentz invariant product of η and p_j implies $\lambda_\eta = \lambda_j$ or $\tilde\lambda_\eta = \tilde\lambda_j$. Combining these two conditions, we find two solutions

$$\eta = \lambda_j \tilde\lambda_i\,, \qquad \eta' = \lambda_i \tilde\lambda_j \tag{2.3}$$

.

[1]Note that the opposite sign convention from (1.6) is used in QCD literature for $[i\ j]$.

JHEP07(2005)025

Now consider the case where the particle i is massless and the particle j is massive. The condition that momentum η is orthogonal to p_i gives $\lambda_\eta = \lambda_i$ or $\tilde{\lambda}_\eta = \tilde{\lambda}_i$. If $\lambda_\eta = \lambda_i$, the orthogonality to p_j reads

$$2\eta \cdot p_j = \lambda_i^a p_{ja\dot{a}} \tilde{\lambda}_\eta^{\dot{a}} = 0 \,, \tag{2.4}$$

hence $\tilde{\lambda}_\eta^{\dot{a}} = \lambda_{ia}\, p_j^{a\dot{a}} = (\lambda_i p_j)^{\dot{a}}$. So the two possible null vectors orthogonal to both p_i and p_j are

$$\eta^{a\dot{a}} = \lambda_i^a (\lambda_i p_j)^{\dot{a}} \,, \qquad \eta'^{a\dot{a}} = (p_j \tilde{\lambda}_i)^a \tilde{\lambda}_i^{\dot{a}} \,. \tag{2.5}$$

The case when i is massive and j is massless is treated analogously.

The last case to consider is when both particles i and j are massive. Here, neither p_i nor p_j is a product of two spinors so the expression for η is not as simple. We use the condition $2\eta \cdot p_i = \lambda_\eta^a \tilde{\lambda}_\eta^{\dot{a}} p_{ia\dot{a}} = 0$ to express $\lambda_{\eta a} = p_{ia\dot{a}} \tilde{\lambda}_\eta^{\dot{a}}$. Putting this into the second orthogonality condition gives a quadratic equation for λ_η

$$\lambda_\eta^a \lambda_\eta^b p_{ia\dot{a}}\, p_{jb\dot{b}} \epsilon^{\dot{a}\dot{b}} = -\langle \eta | p_i \cdot p_j | \eta \rangle = 0 \,. \tag{2.6}$$

This equation has two solutions, $\lambda_\eta^\pm$, which we can find, for example, by setting $\lambda_\eta^a = (1, x)$ and solving the quadratic equation for x. The analogous condition for the positive helicity spinor $\tilde{\lambda}_\eta^{\dot{a}}$

$$\tilde{\lambda}_\eta^{\dot{a}} \tilde{\lambda}_\eta^{\dot{b}} p_{ia\dot{a}}\, p_{jb\dot{b}}\, \epsilon^{ab} = -[\eta | p_i \cdot p_j | \eta] = 0 \tag{2.7}$$

has also two solutions. Altogether, up to scaling, there are two null vectors $\eta = \lambda_\eta \tilde{\lambda}_\eta$ that are orthogonal to p_i, p_j. We do not know of a convenient Lorentz invariant solution to eq. (2.6) and eq. (2.7). This makes the case where both marked particles are massive less tractable than the two simpler cases where at least one of the marked particles is lightlike.

2.1 Derivation of the recursion relations

To construct massive recursion relation for a tree-level n-particle amplitude $\mathcal{A}(p_1, p_2, \ldots, p_n)$, we first mark particles i and j for special treatment and pick one of the two null vectors η satisfying the conditions in eqs. (2.2). Following [4], consider the auxiliary function of one complex variable

$$\mathcal{A}(z) = \mathcal{A}(p_1(z), \ldots, p_i(z), \ldots, p_j(z), \ldots, p_n(z)) \,, \tag{2.8}$$

where $p_k(z) = p_k$ for $k \neq i, j$, and

$$p_i(z) = p_i + z\eta \,, \qquad p_j(z) = p_j - z\eta \,. \tag{2.9}$$

Since η is null and orthogonal to p_i, p_j, the shifted momenta are on-shell

$$p_i(z)^2 = p_i^2 \,, \qquad p_j(z)^2 = p_j^2 \,. \tag{2.10}$$

Equations (2.9) imply that $p_i(z) + p_j(z) = p_i + p_j$, so $\mathcal{A}(z)$ obeys momentum conservation. Hence, it is an on-shell scattering amplitude of particles with complex momenta and can be computed from the usual Feynman rules.

JHEP07(2005)025

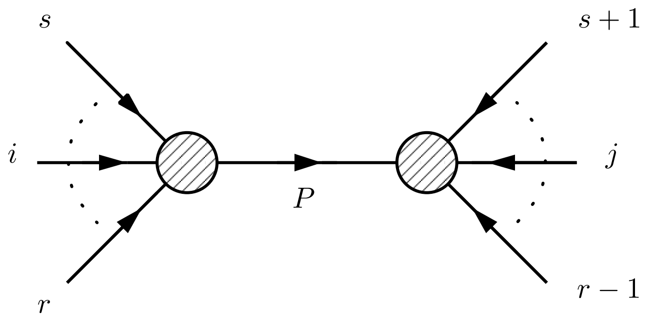

Figure 1: Diagrammatic representation of the recursion relation. Arrows label the momentum flow.

Clearly, the momenta of the external particles are linear functions of z. Notice that the spinors of massless external particles are linear functions of z as well. In the case where both marked particles are massless, there are two possible η's given by eq. (2.3). For $\eta = \lambda_j \tilde{\lambda}_i$, the shift (2.9) is accomplished by

$$\lambda_i(z) = \lambda_i + z\lambda_j \,, \qquad \tilde{\lambda}_j(z) = \tilde{\lambda}_j - z\tilde{\lambda}_i \,. \tag{2.11}$$

The second solution for the shift vector, $\eta' = \lambda_i \tilde{\lambda}_j$, gives

$$\tilde{\lambda}_i(z) = \tilde{\lambda}_i + z\tilde{\lambda}_j \,, \qquad \lambda_j(z) = \lambda_j - z\lambda_i \,. \tag{2.12}$$

Consider now the case when one of the particles, say particle i, is massless and the other particle j is massive. Then eq. (2.5) gives $\eta^{a\dot{a}} = \lambda_i^a (\lambda_i p_j)^{\dot{a}}$. The shift of marked momenta (2.9) is accomplished by

$$\tilde{\lambda}_i^{\dot{a}}(z) = \tilde{\lambda}_i^{\dot{a}} + z(\lambda_i p_j)^{\dot{a}} \,, \qquad p_j^{a\dot{a}}(z) = p_j^{a\dot{a}} - z\lambda_i^a(\lambda_i p_j)^{\dot{a}} \,. \tag{2.13}$$

For $\eta^{a\dot{a}} = (p_j \tilde{\lambda}_i)^a \tilde{\lambda}_i^{\dot{a}}$ there are analogous expressions

$$\lambda_i^a(z) = \lambda_i^a + z(p_j \tilde{\lambda}_i)^a \,, \qquad p_j^{a\dot{a}}(z) = p_j^{a\dot{a}} - z(p_j \tilde{\lambda}_i)^a \tilde{\lambda}_i^{\dot{a}} \,. \tag{2.14}$$

It follows that $\mathcal{A}(z)$ is a rational function of z because at tree level, the scattering amplitude is a rational function of the spinors of massless external particles and of the momenta of massive external particles.

At tree-level the rational function $\mathcal{A}(z)$ can only have simple poles in z coming from internal propagators $1/P(z)^2$. Each propagator divides the external particles into two groups, the particles to the 'left' and to the 'right' of the propagator as illustrated in figure 1. Hence, the momentum $P(z)$ of a propagator is the sum of the momenta of the external particles to the left of the propagator

$$P = p_r + \ldots + p_i + \ldots + p_s \,. \tag{2.15}$$

Momentum $P(z)$ depends on z only if the particles i and j are on opposite sides of the propagator. We choose the particle i to be on the left of the propagator and the particle j to be on the right, as in figure 1. Then

$$P(z) = P + z\eta \,, \tag{2.16}$$

and the propagator is,

$$\frac{1}{P(z)^2 - m^2} = \frac{1}{P^2 - m^2 + 2zP \cdot \eta}\,, \tag{2.17}$$

where m is the mass of the internal particle. The propagator (2.17) has a simple pole at

$$z = -\frac{P^2 - m^2}{2P \cdot \eta}\,. \tag{2.18}$$

For generic external momenta, all internal momenta are different, hence, the locations of all poles are different. It follows that the tree-level amplitude $\mathcal{A}(z)$ has only simple poles as a function of z.

To find the recursion relations, we use the familiar theorem from complex analysis that the sum of residues of a rational function on a Riemann sphere is zero. Applying this to $\mathcal{A}(z)/z$ we express $\mathcal{A}(0)$ as a sum over residues

$$\mathcal{A}(0) = \text{Res}\left(\frac{\mathcal{A}(z)}{z}\right)_{z=0} = -\sum_\alpha \text{Res}\left(\frac{\mathcal{A}(z)}{z}\right)_{z=z_\alpha} - \text{Res}\left(\frac{\mathcal{A}(z)}{z}\right)_{z=\infty}\,, \tag{2.19}$$

where the sum is over all finite poles z_α of the amplitude $\mathcal{A}(z)$. These come from the propagators $1/P^2(z)$ that separate the particles i and j. The residues at finite z are determined by the factorization of the scattering amplitude when the Feynman propagator (2.17) goes on-shell

$$\text{Res}\left(\frac{\mathcal{A}(z)}{z}\right)_{z=z_\alpha} = -\frac{\mathcal{A}_L(z_\alpha)\mathcal{A}_R(z_\alpha)}{P^2 - m^2}\,. \tag{2.20}$$

Here, $\mathcal{A}_L$ and $\mathcal{A}_R$ are the tree-level amplitudes of the particles to the left and to the right of the propagator and z_α is given by eq. (2.18).

Hence, any tree-level scattering scattering amplitude $\mathcal{A} = \mathcal{A}(0)$ can be written in the form

$$\mathcal{A} = +\sum_\alpha \frac{\mathcal{A}_L(z_\alpha)\mathcal{A}_R(z_\alpha)}{P_\alpha^2 - m_\alpha^2} - \text{Res}\left(\frac{\mathcal{A}(z)}{z}\right)_\infty\,, \tag{2.21}$$

where the sum is over all channels α such that the particles i and j are on different sides of the channel, and z_α is given by eq. (2.18). The relation (2.21) is useful for computing scattering amplitudes only if there is an efficient way to determine the boundary contribution $\text{Res}\,(\mathcal{A}(z)/z)_\infty$. The most favourable scenario is when this contribution vanishes. This happens if and only if $\mathcal{A}(z)$ vanishes at infinity,

$$\text{Res}\left(\frac{\mathcal{A}(z)}{z}\right)_\infty = 0 \Leftrightarrow \mathcal{A}(z) \to 0\,, \qquad \text{for} \quad z \to \infty\,, \tag{2.22}$$

in which case, there is a simple recursion relation:

$$\mathcal{A} = \sum_\alpha \mathcal{A}_L(z_\alpha)\frac{1}{P_\alpha^2 - m_\alpha^2}\mathcal{A}_R(z_\alpha) \tag{2.23}$$

that expresses $\mathcal{A}$ in terms of lower-point on-shell scattering amplitudes $\mathcal{A}_L(z_\alpha)$ and $\mathcal{A}_R(z_\alpha)$. The summation in (2.23) runs over all partitions of particles between $\mathcal{A}_L(z_\alpha)$ and $\mathcal{A}_R(z_\alpha)$,

JHEP07(2005)025

such that p_i is on the left, and p_j is on the right of P_α, and also over all helicities h of the intermediate state P_α.

The above considerations apply to the case with massive or massless marked particles. However, for calculations carried out in this paper it will be sufficient to take both marked particles to be massless. In this case, the necessary conditions for the vanishing of the boundary contribution (2.22) put constraints on the possible helicities of the marked particles i and j. We discuss these conditions in section 4.

2.2 Recursion relations: summary

We will use the recursion relations to calculate tree-level scattering amplitudes in Yang-Mills theory coupled to matter fields. The matter fields may be massive or massless and transform in a generic representation of the gauge group. We consider the colour-ordered partial amplitudes $\mathcal{A} = \mathcal{A}(p_1, \ldots, p_n)$, in which the coloured particles come in a definite cyclic order $1, 2, \ldots, n$. These amplitudes are obtained by stripping away the colour factors from the full amplitude. hence, they depend on the kinematic variables, momenta and helicities, p_k and h_k only.

In the remainder of the paper we will take both marked particles to be massless. We shift two massless momenta $p_i = |i\rangle|i]$ and $p_j = |j\rangle|j]$ of the marked particles by $\eta = |j\rangle|i]$, so the shifted momenta are

$$\widehat{p}_i = p_i + z|j\rangle|i]\,, \tag{2.24}$$

$$\widehat{p}_j = p_j - z|j\rangle|i]\,, \tag{2.25}$$

$$\widehat{P} = P + z|j\rangle|i]\,, \tag{2.26}$$

where $P = p_r + \ldots + p_i + \ldots + p_s$ is the momentum of the intermediate particle. For the particles i, j this is equivalent to shifting the spinors

$$|\widehat{i}] = |i]\,, \qquad |\widehat{i}\rangle = |i\rangle + z|j\rangle\,, \tag{2.27}$$

$$|\widehat{j}\rangle = |j\rangle\,, \qquad |\widehat{j}] = |j] - z|i]\,. \tag{2.28}$$

The recursion relation (2.23) written more explicitly is (c.f. figure 1)

$$\mathcal{A}_n(p_1, \ldots, p_n) = \sum_{\text{partitions}} \sum_h \mathcal{A}_L(p_r, \ldots, \widehat{p}_i, \ldots, p_s, -\widehat{P}^h) \frac{1}{P^2 - m_p^2} \times$$

$$\times \mathcal{A}_R(\widehat{P}^{-h}, p_{s+1}, \ldots, \widehat{p}_j, \ldots, p_{r-1})\,, \tag{2.29}$$

where summation is over all partitions of n external particles between $\mathcal{A}_L$ and $\mathcal{A}_R$, such that p_i is on the left, and p_j is on the right, and also over the helicities, h, of the intermediate state. z can be found from the on-shell condition $\widehat{P}^2 = m_p^2$,

$$z = -\frac{P^2 - m_p^2}{2P \cdot \eta} = \frac{P^2 - m_p^2}{\langle j|P|i]}\,. \tag{2.30}$$

When the intermediate state P is a massless particle (e.g. a gluon), we can simplify the spinor products involving $|\widehat{P}\rangle$ and $|\widehat{P}]$ as in ref. [3]:

$$\langle k\ \widehat{P}\rangle = \frac{\langle k|\widehat{P}|i]}{[\widehat{P}\ i]} \equiv \frac{\langle k|P|i]}{\omega}\,, \tag{2.31}$$

JHEP07(2005)025

$$[\widehat{P}\ k] = \frac{\langle j|\widehat{P}|k]}{\langle j\ \widehat{P}\rangle} \equiv \frac{\langle j|P|k]}{\overline{\omega}}\,, \tag{2.32}$$

where ω and $\overline{\omega}$ enter the amplitude always in the combination $\omega\overline{\omega} = \langle j|P|i]$.

For practical computations it is essential that $\mathcal{A}(z)$ vanishes for large z, so that the recursion relations do not have a boundary contribution at infinity. As discussed in section 4, this puts a constraint on the helicities of the particles i and j. For our choice of the shift momentum, $\eta = |j\rangle|i]$, the helicities of the marked particles can take the values,

$$\eta = |j\rangle|i]\ : \qquad (h_i, h_j) = (+,-)\,,\, (+,+)\,,\, (-,-) \tag{2.33}$$

but not $(h_i, h_j) = (-,+)$. Conditions (2.33) are the same for massive and for massless amplitudes.

3. Amplitudes with gluons and massive scalars

In this section we consider scattering amplitudes of gluons with massive complex scalars. These amplitudes are related to amplitudes with *massless* scalars that have D-dimensional momenta. The scalars with D-dimensional momenta P_D can be thought of as massive scalars in 4 dimensions. The D-dimensional on-shell condition, ${P_D}^2 = 0$, gives the 4-dimensional mass-shell equation, ${P_4}^2 = \mu^2$, where the mass term, μ^2, arises from the extra $D-4$ dimensions of momenta.

We will derive amplitudes with 2 scalars and up to 4 gluons with arbitrary helicity configurations. The amplitudes with the same-helicity gluons have been previously derived in [47].

3.1 Primitive vertices

The recursion relations construct n-point amplitudes from on-shell m-point amplitudes with $m < n$. The m-point amplitudes are connected to each other with scalar propagators. Using the recursion relation $n-3$ times gives a representation of the n-point amplitude entirely in terms of the 3-point vertices. Hence, 3-point vertices are the building blocks of the amplitudes in the recursive approach,[2] they will be called the primitive vertices.

The recursion relation reduce the task of computing general amplitudes to the computation of all 3-point primitive vertices. In this paper we consider amplitudes with massless gluons g and massive scalars ϕ. These can be built from the ggg and $\phi g \phi^\dagger$ vertices.

The three-gluon primitive amplitudes can have $--+$ or $++-$ helicity configurations. These are the standard MHV and $\overline{\text{MHV}}$ 3-point on-shell amplitudes

$$\mathcal{A}_3(g_1^-, g_2^-, g_3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\,\langle 31\rangle}\,, \qquad \mathcal{A}_3(g_1^+, g_2^+, g_3^-) = \frac{[12]^3}{[23]\,[31]}\,. \tag{3.1}$$

The gluon momenta k_i are assumed to be complex which ensures that these amplitudes do not vanish on-shell [1, 3].

[2] In particular, this implies that the 4-point vertices in the microscopic lagrangian are not used in the recursive construction of gauge-invariant amplitudes [3].

In order to compute scattering amplitudes of gluons and massive scalars, we need to determine the $\phi g \phi^\dagger$ vertices. To obtain these, we start with the off-shell Feynman vertex of two scalars of mass μ and momenta l_1, l_2, and a single gluon with momentum k,

$$V_3(l_1^+, k^\mu, l_2^-) = \frac{1}{\sqrt{2}}(l_2^\mu - l_1^\mu)\,. \tag{3.2}$$

The $\sqrt{2}$ comes from the normalization conventions used in colour-ordered Feynman rules [49], and the $+$ and $-$ indices are labels for a scalar and an anti-scalar. To derive the desired on-shell amplitudes, $\mathcal{A}_3(l_1^+, k^\pm, l_2^-)$, we contract $V_3(l_1^+, k^\mu, l_2^-)$ with the gluon polarization vector, $\epsilon_\mu^\pm(k,q)$

$$\epsilon^+(k,q)_{a\dot{a}} = \sqrt{2}\frac{q_a\, k_{\dot{a}}}{\langle q\ k\rangle}\,, \qquad \epsilon_\mu^-(k,q) = -\sqrt{2}\frac{k_a\, q_{\dot{a}}}{[k\ q]}\,, \tag{3.3}$$

where $q = |q\rangle|q]$ is an arbitrary reference vector that is not proportional to k. The two independent on-shell vertices immediately follow

$$\mathcal{A}_3(l_1^+, k^+, l_2^-) = \mathcal{A}_3(l_1^-, k^+, l_2^+) = \frac{\langle q_1|l_1|k]}{\langle q_1 k\rangle}\,, \tag{3.4}$$

$$\mathcal{A}_3(l_1^+, k^-, l_2^-) = \mathcal{A}_3(l_1^-, k^-, l_2^+) = -\frac{\langle k|l_1|q_2]}{[q_2 k]}\,. \tag{3.5}$$

We have already noted that the primitive vertices vanish for on-shell real momenta in Minkowski space, but are nonzero for on-shell complex momenta. Indeed, the on-shell conditions, $l_1^2 = l_2^2 = \mu^2$ and $k^2 = 0$, together with the momentum conservation imply that the momentum of the gluon is orthogonal to the momenta of the scalars $k\cdot l_1 = k\cdot l_2 = 0$. For real massless momentum k in Minkowski space, the spinors $k^a, \tilde{k}_{\dot{a}}$ are complex conjugates $k_a^* = \pm\tilde{k}^{\dot{a}}$. Similarly a real massive momentum forms a Hermitian matrix $(l_{a\dot{b}})^\dagger = l_{b\dot{a}}$. Hence for real momenta, the conditions $l_i^{a\dot{a}} k_a \tilde{k}_{\dot{a}} = 0, i = 1,2$ imply $l_i^{a\dot{a}} k_a = l_i^{a\dot{a}}\tilde{k}_{\dot{a}} = 0$ for $i = 1,2$. It follows that the 3-point vertices (3.5-3.4) vanish. For example we have $\mathcal{A}(l_1^+, k^+, l_2^-) \propto q_a l_1^{a\dot{a}}\tilde{k}_{\dot{a}} = 0$.

For complex momenta the spinors $k^a, \tilde{k}^{\dot{a}}$ become independent variables. This additional freedom allows us to take the momenta of the scalars on-shell while keeping the three-valent amplitudes nonzero.

Finally, we note that the primitive amplitudes are gauge-invariant, even though eqs. (3.4)-(3.5) contain explicit q-dependence. Different choice of the reference vector q amounts to a gauge transformation, hence the on-shell amplitudes should not depend on the choice of q by virtue of gauge symmetry. It is easy to see this explicitly e.g. for the $(\phi^+ g^+ \phi^-)$ amplitude. The reference spinor $q^a, a = 1,2$ lives in a two dimensional complex vector space. The spinors q^a and k^a are independent due to the condition $\langle qk\rangle \neq 0$, so we take them as a basis of the vector space. Hence a change in the reference spinor can be parameterized as

$$q'^a = \alpha q^a + \beta k^a\,. \tag{3.6}$$

Changing q, the amplitude becomes

$$\mathcal{A}'(l_1^+, k^+, l_2^-) = \frac{\alpha\ \langle q|l_1|k] + \beta\ \langle k|l_1|k]}{\alpha\langle qk\rangle}\,. \tag{3.7}$$

Here, $\langle k|l_1|k] = -2k\cdot l_1 = \mu^2 - l_2^2 = 0$ is zero by momentum conservation. The remaining α dependence gets cancelled between the numerator and the denominator leaving us with the original amplitude.

It follows that the choice of the reference momenta q_i of the gluons does not affect the amplitude so in principle we could set them to arbitrary values. In the following sections, when using recursion relations to calculate amplitudes with scalars, we will find it convenient to set the reference momentum of a marked gluon in a primitive vertex (3.4) or (3.5) to be the momentum of the other marked gluon.

In the following sections we will calculate tree-level amplitudes of the form

$$\mathcal{A}(\phi^\dagger_{l_1}, g_1, g_2, \ldots, g_m, \phi_{l_2})\,. \tag{3.8}$$

These are the colour-ordered subamplitudes with two massive scalars and m gluons ($2 \leq m \leq 4$) of arbitrary helicities.

When scalars transform in the fundamental representation of the gauge group, the 'string' of fields in the amplitudes must always start and end with the scalar, precisely as in (3.8). Using cyclic symmetry of colour-ordered amplitudes, the scalars in (3.8) can be thought of as adjacent. Scalars in the adjoint representation can appear anywhere in the string, i.e. they do not have to be adjacent. Such amplitudes can also be calculated straightforwardly with our methods.

We will determine all the independent helicity configurations in (3.8), all the remaining configurations can be obtained from those via the following identities:

$$\begin{aligned}
\mathcal{A}_{m+2}(\phi^+_{l_1}, g_1^{h_1}, g_2^{h_2}, \ldots, g_m^{h_m}, \phi^-_{l_2}) &= \mathcal{A}_{m+2}(\phi^-_{l_1}, g_1^{h_1}, g_2^{h_2}, \ldots, g_m^{h_m}, \phi^+_{l_2}) &(3.9)\\
&= (-1)^m\, \mathcal{A}_{m+2}(\phi^-_{l_2}, g_m^{h_m}, \ldots, g_2^{h_2}, g_1^{h_1}, \phi^+_{l_1}) &(3.10)\\
&= \mathcal{A}^*_{m+2}(\phi^-_{l_1}, g_1^{-h_1}, g_2^{-h_2}, \ldots, g_m^{-h_m}, \phi^+_{l_2}) &(3.11)
\end{aligned}$$

where $\ldots$ indicate gluon fields and h_i is the helicity of the i^{th} gluon. Equations (3.10) and (3.11) follow from reflection and parity symmetry of the colour-ordered amplitudes, and (3.9) follows from eqs. (3.4)-(3.5).

3.2 4-point amplitudes

There are two independent helicity amplitudes in this case, the recursion relation (2.29) gives only one term for each of the amplitudes, as illustrated in figure 2.

In the case of two positive helicity gluons, we have:

$$\mathcal{A}_4(l_1^+, 1^+, 2^+, l_2^-) = \mathcal{A}_3(l_1^+, \widehat{1}^+, -\widehat{P}^-)\frac{1}{P^2-\mu^2}\,\mathcal{A}_3(\widehat{P}^+, \widehat{2}^+, l_2^-)\,, \tag{3.12}$$

where we took the marked particles to be the gluons with momenta k_1 and k_2. We set the reference vectors q_1 and q_2 of the two gluons equal to the marked momenta gluon on the opposite side of the diagram

$$q_2 = \widehat{k}_1 = |\widehat{k}_1\rangle|\widehat{k}_1]\,, \qquad q_1 = \widehat{k}_2 = |\widehat{k}_2\rangle|\widehat{k}_2]\,. \tag{3.13}$$

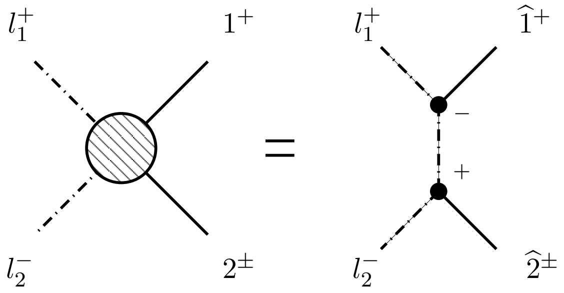

Figure 2: Representation of the 4-point amplitude using the recursion relation with 1 and 2 as the shifted momenta.

We shift the momenta along the vector $\eta = |2\rangle|1]$, so that $|\widehat{1}] = |1]$, $|\widehat{2}\rangle = |2\rangle$. Whence, the amplitude becomes

$$-\frac{\langle 2|l_1|1]\ \langle\widehat{1}|l_2|\widehat{2}]}{\langle 12\rangle^2((l_1+k_1)^2-\mu^2)}\,.$$

Using $l_1 + l_2 + \widehat{k}_1 + \widehat{k}_2 = 0 = l_1 + l_2 + k_1 + k_2$, this can be written as

$$-\frac{\mathrm{tr}(\not{\widehat{k}}_1 \not{l}_1 \not{\widehat{k}}_2 \not{l}_1)}{2\ \langle 12\rangle^2((l_1+k_1)^2-\mu^2)} = -\frac{\langle 12\rangle\ [12](l_1\cdot l_1) - 2(\widehat{k}_1\cdot l_1)(\widehat{k}_2\cdot l_1)}{\langle 12\rangle^2((l_1+k_1)^2-\mu^2)}\,.$$

To get the second expression we used a Fierz identity. The second term in the numerator vanishes, $\widehat{k}_1 \cdot l_1 = 0$. The easiest way to see this is to use momentum conservation in the 3-point vertex, $l_1 + \widehat{k}_1 = \widehat{P}$, and the on-shell conditions $l_1^2 = \widehat{P}^2 = \mu^2$, $\widehat{k}_1^{\,2} = 0$. Alternatively this can be shown with the use of the definition (2.24) of $\widehat{k}$

$$\widehat{k}_1\cdot l_1 = k_1\cdot l_1 - \frac{(l_1+k_1)^2-\mu^2}{2\ \langle 2|l_1|1]}\ \langle 2|l_1|1] = 0\,.$$

This leaves us with the final answer

$$\mathcal{A}_4(l_1^+,1^+,2^+,l_2^-) = \frac{\mu^2\ [12]}{\langle 12\rangle((l_1+k_1)^2-\mu^2)}\,. \tag{3.14}$$

This agrees with the previously known result computed by Bern, Dixon and Kosower [47].

For the amplitude with one positive helicity gluon and one negative helicity gluon the recursion relation in figure 2 yields,

$$\mathcal{A}_4(l_1^+,1^+,2^-,l_2^-) = \mathcal{A}_3(l_1^+,\widehat{1}^+,-\widehat{P}^-)\frac{1}{\widehat{P}^2-\mu^2}\mathcal{A}_3(P^+,\widehat{2}^-,l_2^-)\,. \tag{3.15}$$

Using the same choice for the marked gluons and the reference vectors (3.13) as before gives the result

$$\mathcal{A}_4(l_1^+,1^+,2^-,l_2^-) = -\frac{\langle 2|l_1|1]^2}{\langle 12\rangle\ [12]((l_1+k_1)^2-\mu^2)}\,, \tag{3.16}$$

which we checked against a Feynman diagram calculation.

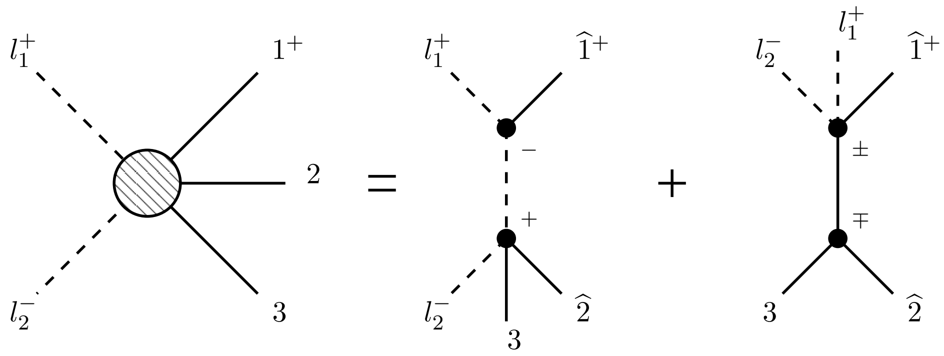

Figure 3: The decomposition of the 5-point amplitude using the recursion relation with 1 and 2 as the shifted momenta.

3.3 5-point amplitudes

The amplitudes with three gluons and a pair of scalars have three independent helicity configurations. As before, we mark the gluons with momenta k_1 and k_2, and pick their reference momenta to be $q_1 = \hat{k}_2$, $q_2 = \hat{k}_1$. The recursion relation is depicted in figure 3. For the amplitude with all gluons of positive helicities, the recursion relations have single non-zero diagram. The diagram with gluon exchange vanishes as the choice of shift vector η implies vanishing of the $\mathcal{A}(\hat{2}^+, 3^+, \hat{p}^\pm)$ $\overline{\text{MHV}}$ amplitude. The amplitude follows immediately:

$$\mathcal{A}_5(l_1^+, 1^+, 2^+, 3^+, l_2^-) = \frac{\mu^2\,[3|(1+2)l_1|1]}{((l_1+k_1)^2-\mu^2)\,\langle 12\rangle\,\langle 23\rangle((l_2+k_3)^2-\mu^2)}\,. \tag{3.17}$$

This is in agreement with the result in [47].

For the case where one of the gluons has negative helicity we have two independent helicity configurations, each of which has two non-zero contributions:

$$\begin{aligned}\mathcal{A}_5(l_1^+, 1^+, 2^+, 3^-, l_2^-) &= -\frac{\langle 3|l_2(1+2)l_1|1]^2}{((l_1+k_1)^2-\mu^2)\,\langle 12\rangle\,\langle 23\rangle((l_2+k_3)^2-\mu^2)\,[3|(1+2)l_1|1]} - \\ &\quad -\frac{\mu^2\,[12]^3}{s_{123}\,[23]\,[3|(1+2)l_1|1]}\,, \end{aligned}\tag{3.18}$$

$$\begin{aligned}\mathcal{A}_5(l_1^+, 1^+, 2^-, 3^+, l_2^-) &= -\frac{\langle 2|l_1|1]^2\,\langle 2|l_2|3]^2}{((l_1+k_1)^2-\mu^2)\,\langle 12\rangle\,\langle 23\rangle((l_2+k_3)^2-\mu^2)\,[3|(1+2)l_1|1]} + \\ &\quad +\frac{\mu^2\,[13]^4}{s_{123}\,[12]\,[23]\,[3|(1+2)l_1|1]}\,. \end{aligned}\tag{3.19}$$

These results are new. Our results (3.17)-(3.19) numerically agree with the much lengthier expressions which we obtained by a direct calculation of the 25 Feynman diagrams.

3.4 6-point amplitudes

We mark gluon momenta 1 and 2, and write down the recursion relation for the 6-point amplitudes with 4 gluons in figure 4.

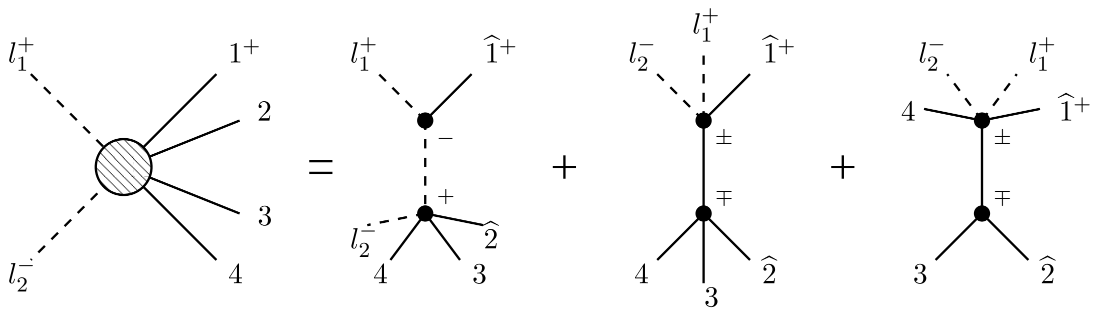

Figure 4: The decomposition of the 6 point amplitude using the recursion relation with 1 and 2 as the shifted momenta.

In the case of all gluons of the same helicity, only the first diagram contributes. We find,

$$\mathcal{A}_6(l_1^+,1^+,2^+,3^+,4^+,l_2^-) = -\frac{\mu^2\,[4|l_2(3+4)(1+2)l_1|1]}{Q_1Q_2Q_3\,\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle}\,,$$

where $Q_1 = ((l_1+k_1)^2-\mu^2)$, $Q_2 = ((l_1+k_1+k_2)^2-\mu^2)$ and $Q_3 = ((l_2+k_4)^2-\mu^2)$. Eq.3.20 is a slightly shorter form of the result given in [47].

Now we compute the remaining independent 6-point amplitudes. There are two amplitudes with one negative helicity gluon:

$$\begin{aligned}
\mathcal{A}_6(l_1^+,1^+,2^+,3^+,4^-,l_2^-) = &+\frac{\big(Q_2\,\langle 4|l_2(1+2+3)l_1|1] - \mu^2(\,\langle 4|l_2 32|1])\big)^2}{Q_1Q_2Q_3\,\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,[4|l_2(3+4)(1+2)l_1|1]}+\\
&+\frac{\mu^2\,[3|(1+2)l_1|1]^3}{Q_1\,\langle 12\rangle\,[34]\,\langle 2|(3+4)(l_1+l_2)l_1|1]\,[4|l_2(3+4)(1+2)l_1|1]}-\\
&-\frac{\mu^2\,\langle 4|2+3|1]^3}{s_{1234}s_{234}\,\langle 23\rangle\,\langle 34\rangle\,\langle 2|(3+4)(l_1+l_2)l_1|1]}
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\mathcal{A}_6(l_1^+,1^+,2^+,3^-,4^+,l_2^-) = &+\frac{(Q_2\,\langle 3|l_1|1] - \mu^2\,[32]\,[21])^2\,\langle 3|l_2|4]^2}{Q_1Q_2Q_3\,\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,[4|l_2(3+4)(1+2)l_1|1]}\\
&+\frac{\mu^2\,[4|(1+2)l_1|1]^4}{Q_1\,\langle 12\rangle\,[34]\,[3|(1+2)l_1|1]}\times\\
&\times\frac{1}{\langle 2|(3+4)(l_1+l_2)l_1|1]\,[4|l_2(3+4)(1+2)l_1|1]}-\\
&-\frac{\mu^2\,\langle 3|2+4|1]^4}{s_{1234}s_{234}\,\langle 4|2+3|1]\,\langle 23\rangle\,\langle 34\rangle\,\langle 2|(3+4)(l_1+l_2)l_1|1]}+\\
&+\frac{\mu^2\,[4|(1+2+3)l_1|1]\,[12]^3}{s_{123}\,\langle 4|2+3|1]Q_3\,[23]\,[3|(1+2)l_1|1]}\,.
\end{aligned} \tag{3.21}$$

These amplitudes agree with the massless MHV-type amplitudes as $\mu^2 \to 0$. There are three independent helicity amplitudes with two negative helicity gluons:

$$\begin{aligned}
\mathcal{A}_6(l_1^+,1^+,2^+,3^-,4^-,l_2^-) = &+\frac{\big(Q_2\,\langle 4|l_2(3+4)l_1|1] - \mu^2\,\langle 4|l_2(3+4)2|1]\big)^2}{Q_1Q_2Q_3\,\langle 12\rangle\,[34]\,\langle 4|l_2(3+4)|2\rangle\,[3|(1+2)l_1|1]}+\\
&+\frac{\mu^2\,\langle 2|l_1|1]^2\,\langle 34\rangle^3}{Q_1\,\langle 12\rangle\,\langle 23\rangle\,\langle 2|(3+4)(l_1+l_2)l_1|1]\,\langle 4|l_2(3+4)|2\rangle}+
\end{aligned}$$

JHEP07(2005)025

$$
\begin{aligned}
&+\frac{[1|l_2 l_1|1]^2\ \langle 34\rangle^3}{s_{1234}s_{234}\ \langle 4|2+3|1]\ \langle 23\rangle\ \langle 2|(3+4)(l_1+l_2)l_1|1]}-\\
&-\frac{\langle 4|l_2(1+2+3)l_1|1]^2\ [12]^3}{Q_3 s_{123}\ \langle 4|2+3|1]\ [4|(1+2+3)l_1|1]\ [23]\ [3|(1+2)l_1|1]}+\\
&+\frac{\mu^2\ [12]^3}{s_{1234}\ [23]\ [34]\ [4|(1+2+3)l_1|1]}
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\mathcal{A}_6(l_1^+,1^+,2^-,3^+,4^-,l_2^-) = &-\frac{\langle 2|l_1|1]^2\ \langle 2|l_1+1|3]^2\ \langle 4|l_2|3]^2}{Q_1Q_2Q_3\ \langle 12\rangle\ [34]\ \langle 4|l_2(3+4)|2\rangle\ [3|(1+2)l_1|1]}-\\
&-\frac{\mu^2\ \langle 24\rangle^4\ \langle 2|l_1|1]^2}{Q_1\ \langle 12\rangle\ \langle 23\rangle\ \langle 34\rangle\ \langle 2|(3+4)(l_1+l_2)l_1|1]\ \langle 4|l_2(3+4)|2\rangle}+\\
&+\frac{[1|l_2 l_1|1]^2\ \langle 24\rangle^4}{s_{1234}s_{234}\ \langle 4|2+3|1]\ \langle 23\rangle\ \langle 34\rangle\ \langle 2|(3+4)(l_1+l_2)l_1|1]}-\\
&-\frac{\langle 4|l_2(1+2+3)l_1|1]^2\ [13]^4}{Q_3 s_{123}\ [12]\ [23]\ [4|(1+2+3)l_1|1]}\times\\
&\times\frac{1}{[3|(1+2)l_1|1]\ \langle 4|2+3|1]}+\\
&+\frac{\mu^2\ [13]^4}{s_{1234}\ [12]\ [23]\ [34]\ [4|(1+2+3)l_1|1]}
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\mathcal{A}_6(l_1^+,1^+,2^-,3^-,4^+,l_2^-) = &-\frac{\langle 2|l_1|1]^2(Q_3\ [4|l_1+1|2\rangle-\mu^2\ [43]\ \langle 32\rangle)^2}{Q_1Q_2Q_3\ \langle 12\rangle\ [34]\ [3|(1+2)l_1|1]\ \langle 4|l_2(3+4)|2\rangle}-\\
&-\frac{\mu^2\ \langle 23\rangle^3\ \langle 2|l_1|1]^2}{Q_1\ \langle 12\rangle\ \langle 34\rangle\ \langle 4|l_2(3+4)|2\rangle\ \langle 2|(3+4)(l_1+l_2)l_1|1]}+\\
&+\frac{[1|l_2 l_1|1]^2\ \langle 23\rangle^3}{s_{1234}s_{234}\ \langle 34\rangle\ \langle 2|(3+4)(l_1+l_2)l_1|1]\ \langle 4|2+3|1]}\\
&-\frac{[1|(2+3)l_1|1]^2 2}{s_{123}Q_3\ [12]\ [23]\ [3|(1+2)l_1|1]}\times\\
&\times\frac{[1|(2+3)l_2|4]^2}{[4|(1+2+3)l_1|1]\ \langle 4|2+3|1]}+\\
&+\frac{\mu^2\ [14]^4}{s_{1234}\ [12]\ [23]\ [34]\ [4|(1+2+3)l_1|1]}\,.
\end{aligned}
\tag{3.24}
$$

All of the above six-point amplitudes have been checked numerically against an independent calculation of the 220 Feynman diagrams.

4. Vanishing of $A(z)$ at infinity

The recursion relations are valid as long as $\mathcal{A}(z) \to 0$ for $z \to \infty$. This is the case only for some choices of marked particles. Indeed, for the marked gluons i, j with

$$
|\hat{i}\rangle = |i\rangle + z|j\rangle\,, \qquad |\hat{j}] = |j] - z|j]\,,
\tag{4.1}
$$

it was observed in [3] that $\mathcal{A}(z) \to 0$ for the helicity assignments of gluons $(h_i, h_j) = (+,+), (+,-), (-,-)$. This has been extended to fermions in [19, 20]. A direct diagrammatic proof for gluon amplitudes in the $(+,-)$ case was given in [4]. Here, we present an

JHEP07(2005)025

argument valid for all three helicity assignments of marked gluons putting no restrictions on the remaining partons.[3] We also consider the case where one or both of the marked particles are massless fermions.

4.1 The $(+,-)$ case

We start with the $(+,-)$ case for which we will show that all Feynman diagrams contributing to $\mathcal{A}(z)$ vanish at infinity. In a given Feynman diagram, the z-dependence flows along a unique path of Feynman propagators and vertices. The Feynman diagrams with most dangerous z-dependence are those in which all vertices along this path are trivalent. These vertices are linear in momentum so they each contribute a factor of z. Each of the propagators gives a factor of $1/z$. A path made of r propagators has $r+1$ vertices. Hence, the propagators and vertices go at infinity as $z^{r+1}/z^r = z$. For fermions, the propagator $(\not{p}+m)/(p^2-m^2)$ goes like a constant at infinity, but so do the vertices containing fermions. Each of the fermion propagators increases the above estimate by a factor of z and each of the fermion vertices decreases the estimate by z. For f fermion propagators, there are at least $f+1$ fermion vertices along the path of the z-dependence, hence the above bound on the Feynman diagram gets improved by at least a factor of $1/z$.

The remaining pieces of the Feynman diagram that depend on z are the polarization vectors of the marked gluons. Consider marked gluons of opposite helicity, so that we can have either g_i^+, g_j^-, or g_i^-, g_j^+. The spinors $\lambda_i(z)$ and $\tilde{\lambda}_j(z)$ are linear in z, eq. (2.11), while $\tilde{\lambda}_i(z), \lambda_j(z)$ are independent of z. It follows from eq. (3.3) that for the helicity configuration g_i^+, g_j^-, the polarization vectors of both gluons go as $1/z$ at infinity. Altogether Feynman diagrams vanish at infinity as $\mathcal{O}(z/z^2) = \mathcal{O}(1/z)$.

In the opposite case, g_i^-, g_j^+, gluon polarization vectors grow as z at infinity and individual Feynman diagrams go as $\mathcal{O}(z^3)$ leading to nontrivial boundary contributions to the recursion relations for $\mathcal{A}(z)$.

If one of the marked particles is a gluon and the other a massless fermion, we can proceed analogously. Now the propagators and vertices contribute a factor of at worst z^0 because of the fermions. For helicity assignments $(h_i, h_j) = (+1, -1/2), (+1/2, -1)$, the gluon polarization vector goes like $1/z$ and the fermion polarization spinors $u^-(\widehat{p}_j) = \lambda_j^a$ and $u^+(\widehat{p}_i) = \tilde{\lambda}_i^{\dot{a}}$ are independent of z. Hence the shifted amplitude vanishes at infinity for the assignment $(h_i, h_j) = (+,-)$.

When both of the marked particles are fermions, so that the fermion propagates right through the diagram, all vertex, propagator and wavefunction factors are independent of z (in the $z \to \infty$ limit), leading to a non-vanishing contribution as $z \to \infty$. The naive Feynman diagram discussion presented here indicates that marking a fermion-antifermion pair of the same flavour is not allowed. A more careful argument appealing to MHV rules [20] shows that only adjacent fermion and antifermion of the same flavour cannot be marked.

[3] For the $(+,+)$ case, our proof works for the amplitudes that have two extra negative helicity gluons. However, see the end of this section for a heuristic argument, which removes this assumption.

If two different flavour fermions of opposite helicity are marked, there will be at least one additional gluon propagator yielding an extra factor of $1/z$ thereby eliminating the contribution as $z \to \infty$.[4]

4.2 The $(+,+)$ and $(-,-)$ cases

The $(h_i, h_j) = (+,+)$ case cannot be treated by counting powers of z in *individual* Feynman diagrams and requires a more elaborate argument along the lines of [22]. The argument works for amplitudes that, besides gluons g_i^+, g_j^+ have two extra negative helicity gluons g_k^-, g_l^-, hence in particular it works for all QCD gluon amplitudes since at tree level, the gluon amplitudes always have at least two gluons of both helicities.

Consider the function $\mathcal{A}(\lambda_m(z), \tilde{\lambda}_m(z))$ constructed from the scattering amplitude $\mathcal{A}$ by shifting the spinors

$$\begin{aligned} |i\rangle &= |i\rangle + z|k\rangle + z|l\rangle \\ |k] &= |k] - z|i] \\ |l] &= |l] - z|i] \end{aligned} \tag{4.2}$$

while keeping $|i]$, $|k\rangle$ and $|l\rangle$ unshifted. The shifts (4.2) maintain momentum conservation, so $\mathcal{A}(z)$ can be computed from usual Feynman diagrams.

Note that the function $\mathcal{A}(z)$ vanishes at infinity. As in the previous case, we prove this by studying the most dangerous Feynman diagrams. The z-dependence flows along a 'three-legged path' of Feynman propagators and vertices, with each leg ending at one of the external gluons i, k, l This path is illustrated in figure 5. For a path consisting of r propagators, there are $r+1$ vertices. For large z, the propagators and vertices give a $z^{r+1}/z^r = z$ contribution to the z-dependence of $\mathcal{A}(z)$. The polarization vectors of each of the gluons give a factor of $1/z$ so the function $\mathcal{A}(z)$ vanishes for large z as $\mathcal{O}(1/z^2)$.

Hence, following the discussion of recursion relations in section 2, we have

$$\mathcal{A} = \sum_{\alpha,h} \mathcal{A}_L(z_\alpha, -P_\alpha(z_\alpha)^h) \frac{1}{P_\alpha^2 - m_\alpha^2} \mathcal{A}_R(z_\alpha, P_\alpha(z_\alpha)^{-h}) \,, \tag{4.3}$$

where the summation is over partitions of the particles into two sets L, R such that $i \in L$ and at least one of j, k is in R, and also over helicities of the intermediate state P_α.

The amplitudes $\mathcal{A}_L, \mathcal{A}_R$ are evaluated at

$$z_\alpha = \frac{P_\alpha^2 - m_\alpha^2}{\langle k|P_\alpha|i] + \langle l|P_\alpha|i]} \,. \tag{4.4}$$

Similar recursion relations with three or more marked particles have been recently considered in [38, 21, 22].

We will use this expression for $\mathcal{A}$ in an induction proof of the recursion relations with $(h_i, h_j) = (+,+)$. We consider the function $\mathcal{A}(y)$ with

$$|i\rangle = |i\rangle + y|j\rangle \qquad |j] = |j] - y|i] \tag{4.5}$$

[4] The last argument is invalidated in theories with scalar fields ϕ^{AB}, (e.g. in the $\mathcal{N} = 4$ Yang-Mills) which couple two fermions of different flavours, Λ_A and Λ_B. In these theories one cannot mark two adjacent fermions of different flavours and same helicity, $\Lambda_A^\pm$ and $\Lambda_B^\pm$.

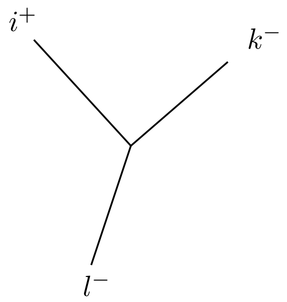

Figure 5: In a Feynman diagram with marked particles i, k, l, the z dependence flows along a unique three-legged path of propagators and vertices.

and estimate the y dependence of $\mathcal{A}(y)$ with the help of eq. (4.3). For a partition α such that i, j are both in L, neither P_α nor z_α depend on y. So the entire y dependence comes from $\mathcal{A}_L$. But $\mathcal{A}_L$ has fewer particles than $\mathcal{A}$ so by induction hypothesis it vanishes as y goes to infinity.

The other case with $i \in L, j \in R$ is more subtle. We have $P_\alpha(y) = P_\alpha + y\lambda_j\tilde{\lambda}_i$ so z_l depends linearly on y

$$z_\alpha = \frac{P_\alpha^2 - y \langle j|P_\alpha|i] - m_\alpha^2}{\langle k|P_\alpha|i] + \langle l|P_\alpha|i]} \,. \tag{4.6}$$

We represent the on-shell amplitudes $\mathcal{A}_L, \mathcal{A}_R$ as a sum of Feynman diagrams. The y dependence flows along a four-legged tree with each leg of the tree ending at one of the gluons i, j, k, l. A tree composed of r propagators (one of which is the propagator connecting $\mathcal{A}_L$ and $\mathcal{A}_R$), contains $r+1$ vertices. The most dangerous contribution comes from Feynman diagrams in which the $r+1$ vertices are three-point vertices that each give a factor of y. Hence, the propagators give a factor of $1/y^r$ and the vertices a factor of y^{r+1}.

The polarization vectors of the gluons i, k, l each give a factor of $1/y$ and that of the gluon j a factor of y, so altogether we have $1/y^2$ coming from the polarization vectors. There are two more 'external' particles with y dependent momentum. These come from the ends of the internal line connecting $\mathcal{A}_L$ and $\mathcal{A}_R$. They contribute $\sum_h \epsilon^h\epsilon^{-h}$ which, in a convenient gauge, goes like a constant at infinity. For example for a gluon we have

$$\sum_{h=\pm} \epsilon_\mu^h \epsilon_\nu^{-h} = g_{\mu\nu} - \frac{P_\mu q_\nu + P_\nu q_\mu}{P \cdot q} \,, \tag{4.7}$$

where $P = P_\alpha(y)$ and q is the reference momentum of the internal gluon. For large y, this is independent of y. Hence, $\mathcal{A}(y)$ vanishes as $\mathcal{O}(y^{r+1}/(y^r y^2)) = \mathcal{O}(1/y)$ for large y.

In this proof, we have assumed that the amplitude has besides the two marked positive helicity gluons i, j also two negative helicity gluons k, l. However we believe that this restriction is not necessary. Heuristically, one can argue as follows. Consider any amplitude $\mathcal{A}$ without imposing the restriction that it contains two negative helicity gluons. We can construct an 'auxiliary' amplitude $\mathcal{A}'$ which, besides external particles present in $\mathcal{A}$ also has two extra negative helicity gluons k, l. In the previous argument, we demonstrated that $\mathcal{A}'(y)$ with marked gluons i, j vanishes for large y as $\mathcal{O}(1/y)$. One can imagine taking soft gluon limit $p_l, p_k \to 0$ to recover the original amplitude. Taking the soft limit gives back a universal factor, times an amplitude with the gluon removed. The universal factor depends on the momentum of the soft gluon and of the possible adjacent particles. Hence, as long as the 'auxiliary' gluons k, l are not adjacent to the marked gluons i, j, their soft factors are y independent. So the y asymptotic of the amplitude $\mathcal{A}$ with gluons k, l removed should be the same as that of $\mathcal{A}'$.

5. Summary

In this paper we have constructed tree-level recursion relations valid in any quantum field theory which naturally incorporate massive particles. As an application, we focus on scalar particles. However, there is no obstacle in applying our general method to massive particles with spin. With the general recursion relations in place, one may be able to avoid in future all Feynman-diagrams calculations of nontrivial tree amplitudes.

As a first application of these recursion relations, we have derived expressions for scattering amplitudes involving a pair of massive scalars and up to four gluons. Explicit results are given in section 3. These compact analytic expressions for amplitudes may be useful in deriving the one-loop six gluon amplitudes in QCD using the unitarity method of Bern et al [46, 47].

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